Integral as Area and Riemann Sums UCLA AP Readiness, Nov 15

We have learned about **limits**. We have learned that the **derivative** measures the **slope** of the curve at any point by using limits. The limit definition of the derivative is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

The definite **integral** represents the area under a curve. It can also be solved using limits.

We begin with Riemann Sums to approximate area under a curve.

$$S = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \qquad x_{i-1} \le x_i^* \le x_i$$

Left Riemann sum: $x_i^* = x_{i-1}$ for all i

Right Riemann sum: $x_i^* = x_i$ for all i

Middle Riemann sum: $x_i^* = \frac{1}{2}(x_i + x_{i-1})$ for all i

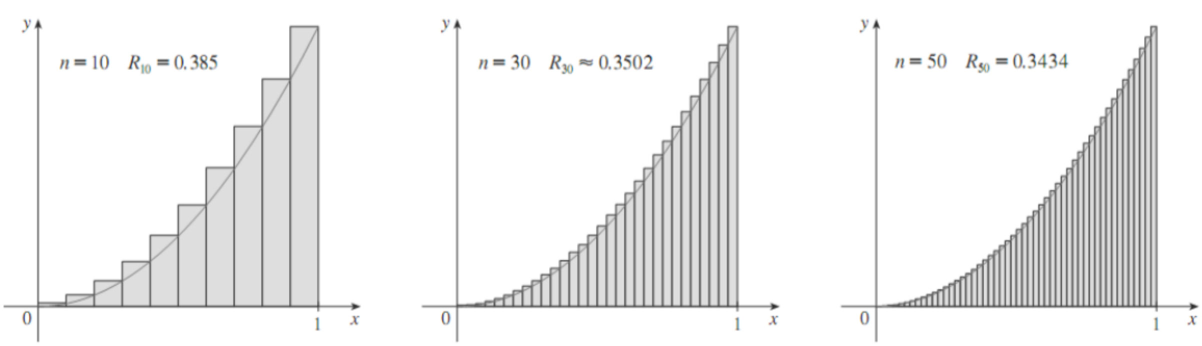
Approximate the area under the curve $f(x)=x^2$ on the interval [0,4] using:

1. Left Rectangles with four subintervals: LRAM=
2. Right Rectangles with four subintervals RRAM=
3. Midpoint Rectangles with four subintervals MRAM=
4. Trapezoids with four subintervals TRAP=

Limit Definition of the Integral

If we desire better approximations of the area, we could partition our area into smaller subintervals using more rectangles. The following chart shows the areas of the same region S, using n rectangles of equal width using both the left-endpoint and right-endpoint methods.

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3282500	0.3383500
1000	0.3328335	0.3338335



$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The Fundamental Theorem of Calculus: Part I

Suppose that *f* is bounded on the interval [*a*,*b*], and that *F* is an antiderivative of *f*, i.e., *F'* = *f*.

Then $\int_a^b f(x)dx = F(a) - F(b)$

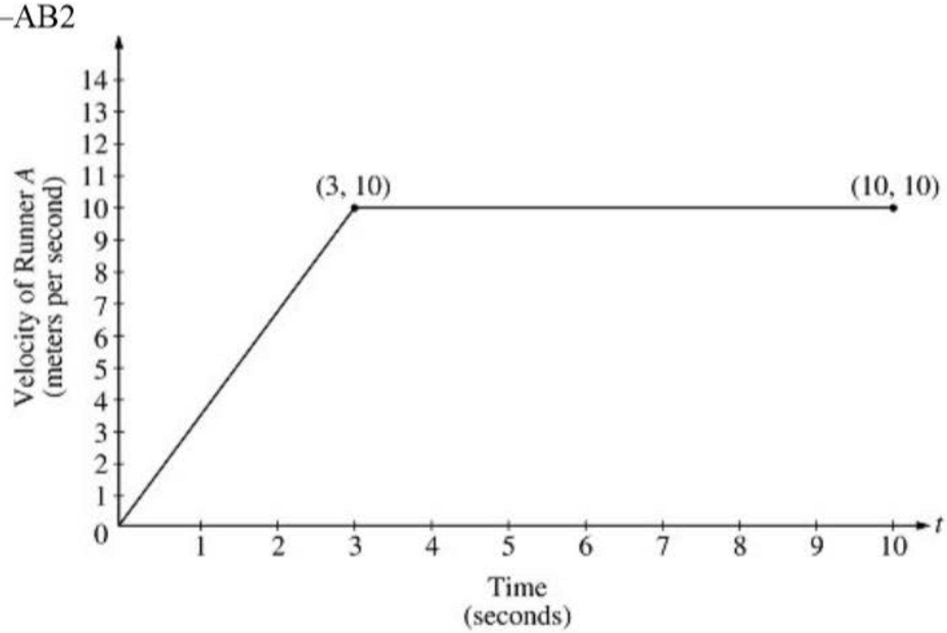
Find the exact area under f(x)=x^2 on [0,4] the curve using the FTC.

Verify the result using your calculator's ability to integrate. On a TI it is "Math 9".

11. If $f(x)$ is represented by the table below, approximate $\int_1^{9.6} f(x)dx$ using left-endpoint, right-endpoint, midpoint, and trapezoidal approximations. Label each one. Use as many subintervals as the data allows.

x	1	2.5	4	6	8	8.8	9.6	10.4
$f(x)$	4	3	1	3	5	6	4	7

12. 2000—AB2



Two runners, *A* and *B*, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner *A*. The velocity, in meters per second, of Runner *B* is given by the function *v* defined by $v(t) = \dfrac{24t}{2t+3}$.

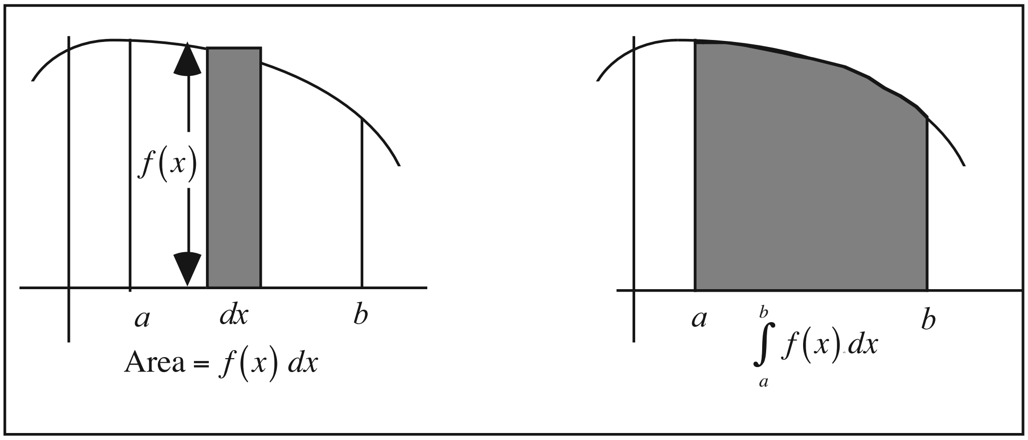
(a) Find the velocity of Runner *A* and the velocity of Runner *B* at time $t = 2$ seconds. Indicate units of measure.

(b) Find the acceleration of Runner *A* and the acceleration of Runner *B* at time $t = 2$ seconds. Indicate units of measure.

(c) Find the total distance run by Runner *A* and the total distance run by Runner *B* over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

The Definite Integral as Area - Classwork

Instead of using the expression "the area under the curve $f(x)$ between $x = a$ and $x = b$, we will now denote a shorthand to represent the same thing. We will use what is called "a definite integral." The definite integral sign is the same as the indefinite integral sign ($\int$) but will contain two limits of integration. The form is as follows: $\int_a^b f(x)\,dx$. While this does not seems to make much sense, there is a reason for it. The $f(x)$ represents the height of any one rectangle while the dx represents the width of any one rectangle. So $f(x)\,dx$ means the area of any one rectangle. The integral represents the sum of an infinite number of these rectangles. The *a* represents the starting place for these rectangles while the *b* represents the ending place for these integrals.



The area of one rectangle = $f(x)\,dx$

The sum of an infinite number of rectangle areas. Each rectangle is infinitely thin.

When $a < b$, we are determining the area under the curve from left to right. In that case, our *dx* is a positive number. If $f(x)$ is above the axis, then $\int_a^b f(x)\,dx$ will be a positive number.

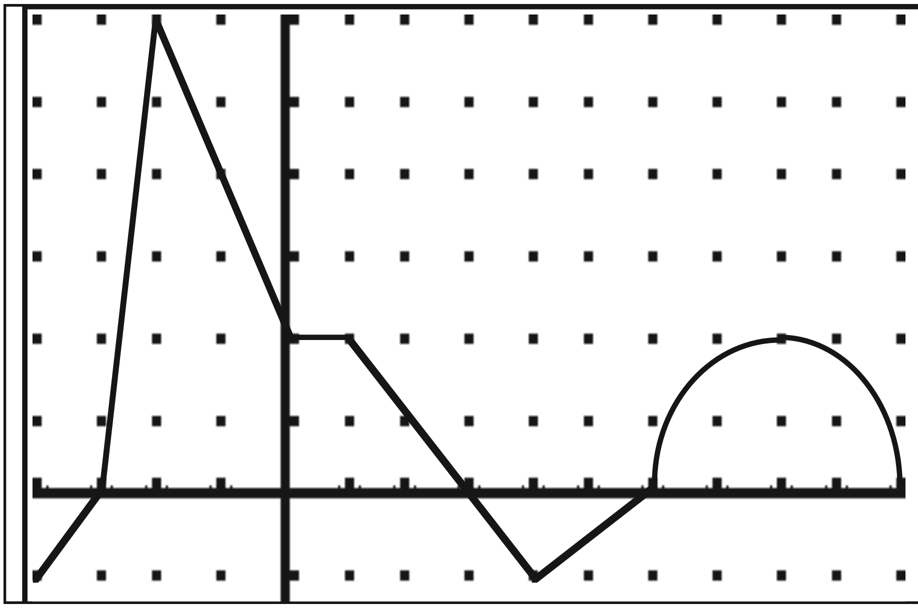
When $b < a$, we are determining the area under the curve from right to left. In that case, our *dx* is a negative number. If $f(x)$ is above the axis, then $\int_a^b f(x)\,dx$ will be a negative number. This can be summarized below:

	$f(x) > 0$ (curve above axis)	$f(x) < 0$ (curve below axis)
$dx > 0$ (left to right) ($a < b$)	$\int_a^b f(x)\,dx > 0$ (Area positive)	$\int_a^b f(x)\,dx < 0$ (Area negative)
$dx < 0$ (right to left) ($b < a$)	$\int_a^b f(x)\,dx < 0$ (Area negative)	$\int_a^b f(x)\,dx > 0$ (Area positive)

Furthermore, there are three more rules which will make sense to you:

1. $\int_a^a f(x)\,dx = 0$ - If we start at *a* and end at *a*, there is no area.
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ - From *a* to *b* gives an area. From *b* to *a* gives the negative of this area.
3. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ - Total the area from *a* to *b,* add area from *b* to *c* = the area from *a* to *c*.

Example) Below you are given the graph of $f(x)$ formed by lines and a semi-circle. Find the definite integrals.



This is the graph of $f(x)$

1. $\int_{4}^{4} f(t)\,dt$

2. $\int_{0}^{1} f(t)\,dt$

3. $\int_{1}^{3} f(t)\,dt$

4. $\int_{0}^{3} f(t)\,dt$

5. $\int_{3}^{6} f(t)\,dt$

6. $\int_{6}^{3} f(t)\,dt$

7. $\int_{0}^{6} f(t)\,dt$

8. $\int_{6}^{10} f(t)\,dt$

9. $\int_{10}^{6} f(t)\,dt$

10. $\int_{0}^{10} f(t)\,dt$

11. $\int_{10}^{0} f(t)\,dt$

12. $\int_{-1}^{0} f(t)\,dt$

13. $\int_{-3}^{0} f(t)\,dt$

14. $\int_{0}^{-3} f(t)\,dt$

15. $\int_{-4}^{-3} f(t)\,dt$

16. $\int_{-4}^{0} f(t)\,dt$

17. $\int_{-4}^{10} f(t)\,dt$

18. $\left|\int_{0}^{10} f(t)\,dt\right|$

19. $\int_{0}^{10} \left|f(t)\right| dt$

20. $\left|\int_{-4}^{10} f(t)\,dt\right|$

21. $\int_{10}^{-4} \left|\,2f(t)\,\right| dt$

Suppose $\int_{-2}^{5} f(x)\,dx = 18$, $\int_{-2}^{5} g(x)\,dx = 5$, $\int_{-2}^{5} h(x)\,dx = -11$ and $\int_{-2}^{8} f(x)\,dx = 0$, find

22. $\int_{-2}^{5} \left(f(x)+g(x)\right) dx$

23. $\int_{-2}^{5} \left[f(x)+g(x)-h(x)\right] dx$

24. $\int_{5}^{-2} 4g(x)\,dx$

25. $\int_{-2}^{5} \left(g(x)+2\right) dx$

26. $\int_{-2}^{5} \left(f(x)-6\right) dx$

27. $\int_{0}^{7} h(x-2)\,dx$

28. $\int_{-4}^{3} g(x+2)\,dx$

29. $\int_{5}^{8} f(x)\,dx$

30. $\int_{1}^{8} \left[f(x-3)+3\right] dx$